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1999 J. Phys. A: Math. Gen. 32 8095

(http://iopscience.iop.org/0305-4470/32/46/308)

J. Phys. A: Math. Gen. **32** (1999) 8095–8110. Printed in the UK PII: S0305-4470(99)05015-5

Planar harmonic polynomials of type B

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Received 9 June 1999

Abstract. The hyperoctahedral group acting on $\mathbb{R}^N$ is the Weyl group of type B and is associated with a two-parameter family of differential-difference operators $\{T_i : 1 \leqslant i \leqslant N\}$. These operators are analogous to partial derivative operators. This paper finds all the polynomials h on $\mathbb{R}^N$ which are harmonic, $\Delta_B h = 0$ and annihilated by T_i for $i > 2$, where the Laplacian $\Delta_B = \sum_{i=1}^N T_i^2$. They are given explicitly in terms of a novel basis of polynomials, defined by generating functions. The harmonic polynomials can be used to find wavefunctions for the quantum many-body spin Calogero model.

1. Introduction

For each finite reflection group there are families of invariant inner products on the space of polynomials, defined by an algebraic expression, and by integration with respect to invariant weight functions on the sphere or on all of Euclidean space. These inner products essentially coincide on the polynomials harmonic with respect to the associated Laplacian. In this paper we study certain specific explicit harmonic polynomials associated with the hyperoctahedral group on $\mathbb{R}^N$. For the reflection groups on $\mathbb{R}^2$ all the harmonic polynomials are known as expressions in Jacobi polynomials. Also, orthogonal bases whose elements are of generalized Hermite (or Laguerre) type, for the Gaussian weight function have been determined by means of a construction using nonsymmetric Jack polynomials. However, an orthogonal basis for the weight functions on the sphere (and the ball or the simplex) has not yet been explicitly found. Here we consider the analogue of ordinary harmonic polynomials in two variables, but harmonic for the N-variable Laplacian Δ_B; that is, polynomials annihilated by T_i for $i > 2$, where $\{T_i : 1 \leqslant i \leqslant N\}$ is the set of differential-difference operators of type B and $\Delta_B = \sum_{i=1}^N T_i^2$. In previous work the author introduced a family of polynomials (the 'p basis') for which it is easy to write down polynomials annihilated by any desired subset of $\{T_i : 1 \leqslant i \leqslant N\}$. In this study it is important to select a set of polynomials for which the harmonic polynomials have 'nice' coefficients. For example, coefficients of hypergeometric type (Pochhammer symbols) are 'nice'. We introduce a set of polynomials which are in the $\boldsymbol{Q}$-span of the p basis (coefficients independent of the parameters) and which allow nice expressions. The definition is given by means of generating functions.

For the harmonic polynomials we will find the values at a special point, $(1, 1, \ldots, 1)$, the leading coefficients and the L^2 norms; the first two are in closed form using ${}_2F_1$ and ${}_3F_2$ summations, respectively, the last is a balanced ${}_4F_3$ sum. Finally, there is a discussion of the important applications of the polynomials, especially as wavefunctions for spin Calogero quantum many-body models.

2. Overview of the results

The finite group of orthogonal transformations which is generated by sign changes and permutation of coordinates on $\mathbb{R}^N$ is called the Weyl group of type B, and will be denoted by W_N. There is a family of measures associated with W_N in a natural way: for positive parameters k, k_1 let

$$\mathrm{d}\mu\,(x; k, k_1) = \prod_{i=1}^{N} |x_i|^{2k_1} \prod_{1\leqslant i<j\leqslant N} |x_i^2 - x_j^2|^{2k} \exp\left(-\frac{|x|^2}{2}\right) \mathrm{d}x. \qquad (2.1)$$

Analysis for functions related to these measures depends on the differential-difference operators constructed by the author [2]. Note that the polynomial terms in $\mathrm{d}\mu$ correspond to linear functions vanishing on the reflecting hyperplanes of the Coxeter group W_N. The reflections in W_N consist of $\{\sigma_i : 1 \leqslant i \leqslant N\}$ and $\{\sigma_{ij}, \tau_{ij} : 1 \leqslant i < j \leqslant N\}$ defined by

$$x\sigma_i = (x_1, \ldots, \overset{i}{-x_i}, \ldots, x_N)$$
$$x\sigma_{ij} = (x_1, \ldots, \overset{i}{x_j}, \ldots, \overset{j}{x_i}, \ldots, x_N)$$
$$x\tau_{ij} = (x_1, \ldots, \overset{i}{-x_j}, \ldots, \overset{j}{-x_i}, \ldots, x_N).$$

For notational convenience $\sigma_{ji} = \sigma_{ij}$ and $\tau_{ji} = \tau_{ij}$. We use the same symbols to indicate the action on functions, for example, $\sigma_{ij} f(x) := f(x\sigma_{ij})$. The differential-difference ('Dunkl') operators of type B (associated with W_N) are

$$T_i := \frac{\partial}{\partial x_i} + k_1 \frac{1-\sigma_i}{x_i} + k \sum_{j\neq i} \left\{ \frac{1-\sigma_{ij}}{x_i - x_j} + \frac{1-\tau_{ij}}{x_i + x_j} \right\} \qquad 1 \leqslant i \leqslant N.$$

They are homogeneous of degree -1 on polynomials and commute, $T_i T_j = T_j T_i$. The Laplacian operator is $\Delta_B := \sum_{i=1}^{N} T_i^2$.

In this paper we will determine all polynomials h on $\mathbb{R}^N$ which satisfy $\Delta_B h = 0$ and $T_i h = 0$ for all $i > 2$. In the standard case, $k = 0 = k_1$, this implies that h depends only on x_1, x_2; this explains the name 'planar'.

Definition 1. *The set of polynomials* $\{\phi_{n,j}, \psi_{n,j} : 0 \leqslant j \leqslant n = 0, 1, 2, \ldots\}$ *is defined by*

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \phi_{n,j}(x) s^j t^n = F_0 + F_1$$
$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \psi_{n,j}(x) s^j t^n = x_1(F_0 + sF_1) - x_1 F_1$$

in terms of the generating functions ($x \in \mathbb{R}^N$*, and absolute convergence holds for* $|s| < \frac{4}{3}$ *and* $|t| < \min(1/x_i^2 : 1 \leqslant i \leqslant N)/3$*):*

$$F_0(x; s, t) = \frac{1 - st(x_1^2 + x_2^2) + t^2 x_1^2 x_2^2}{(1 - 2stx_1^2 + t^2 x_1^4)(1 - 2stx_2^2 + t^2 x_2^4)} \prod_{i=1}^{N} (1 - 2stx_i^2 + t^2 x_i^4)^{-k}$$
$$F_1(x; s, t) = \frac{t(x_1^2 - x_2^2)}{(1 - 2stx_1^2 + t^2 x_1^4)(1 - 2stx_2^2 + t^2 x_2^4)} \prod_{i=1}^{N} (1 - 2stx_i^2 + t^2 x_i^4)^{-k}.$$

In each case, the first and second terms on the right-hand side produce the basis functions with $n + j = 0 \bmod 2$ and $n + j = 1 \bmod 2$, respectively. The polynomials $\phi_{n,j}$ and $\psi_{n,j}$ are of degrees $2n$ and $2n + 1$ in x, respectively, for $0 \leqslant j \leqslant n$. Because of the invariance of Δ_B

under the subgroup W_2 acting on the first two variables, there is a basis of planar harmonic polynomials in which all the monomials have the same parities of the exponents of x_1, x_2; of course they are even in each of the remaining variables.

2.1. The harmonic polynomials

The basis elements are labelled $h_{n,0}$ and $h_{n,1}$, where $x_1^n x_2^\varepsilon$ is the term in h with the highest power of x_1 and $\varepsilon = 0$ or 1. The formulae depend on the mod 4 residues:

$$h_{4n,0} = \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{1}{2} + 2n)_j (\frac{1}{2})_j}{((N-1)k + k_1 + \frac{1}{2} + n)_j (Nk + n + 1)_j} \phi_{2n,2j}$$

$$h_{4n+2,0} = \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)_j (\frac{1}{2})_j}{((N-1)k + k_1 + \frac{3}{2} + n)_j (Nk + n + 2)_j} \phi_{2n+1,2j}$$

$$h_{4n+1,1} = x_1 x_2 \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)_j (\frac{1}{2})_j}{((N-1)k + k_1 + \frac{3}{2} + n)_j (Nk + n + 1)_j} \phi_{2n,2j}$$

$$h_{4n+3,1} = x_1 x_2 \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{5}{2} + 2n)_j (\frac{1}{2})_j}{((N-1)k + k_1 + \frac{5}{2} + n)_j (Nk + n + 2)_j} \phi_{2n+1,2j}.$$

The following are of mixed parity:

$$h_{4n+1,0} = \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)_j (\frac{1}{2})_j}{((N-1)k + k_1 + \frac{3}{2} + n)_j (Nk + n + 1)_j} \psi_{2n,2j} + \sum_{j=1}^{n} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)_{j-1} (\frac{1}{2})_j}{((N-1)k + k_1 + \frac{3}{2} + n)_{j-1} (Nk + n + 1)_j} \psi_{2n,2j-1}$$

and

$$h_{4n+3,0} = \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{5}{2} + 2n)_j (\frac{1}{2})_j}{((N-1)k + k_1 + \frac{3}{2} + n)_j (Nk + n + 2)_j} \psi_{2n+1,2j} + \sum_{j=1}^{n+1} \frac{(2(N-1)k + k_1 + \frac{5}{2} + 2n)_{j-1} (\frac{1}{2})_j}{((N-1)k + k_1 + \frac{3}{2} + n)_j (Nk + n + 2)_{j-1}} \psi_{2n+1,2j-1}.$$

There are two other basis elements, defined as $h_{4n,1} := \sigma_{12} h_{4n+1,0}$ and $h_{4n+2,1} := \sigma_{12} h_{4n+3,0}$. Furthermore, $\sigma_{12} h_{4n,0} = h_{4n,0}, \sigma_{12} h_{4n+1,1} = h_{4n+1,1}$, and $\sigma_{12} h_{4n+2,0} = -h_{4n+2,0}, \sigma_{12} h_{4n+3,1} = -h_{4n+3,1}$ from the obvious symmetry properties of the generating functions.

2.2. Action of T_i

Since Δ_B commutes with each T_i the action of T_1 or T_2 on any of the polynomials $h_{n,\varepsilon}$ produces a polynomial annihilated by Δ_B and T_i, $i > 2$, that is, a scalar multiple of another polynomial of this family. Specifically, the results are (for $n = 0, 1, 2, \ldots$):

$$T_1 h_{4n,0} = 2((N-1)k + n) h_{4n-1,0}$$
$$T_1 h_{4n+3,0} = -2((N-2)k + k_1 + n + \tfrac{1}{2}) h_{4n+2,0}$$
$$T_1 h_{4n+2,0} = 2(Nk + n + 1) h_{4n+1,0}$$
$$T_1 h_{4n+1,0} = 2((N-1)k + k_1 + n + \tfrac{1}{2}) h_{4n,0}$$

$$T_2 h_{4n+3,0} = 2(Nk+n+1)h_{4n+1,1}$$
$$T_2 h_{4n+1,1} = 2((N-2)k+k_1+n+\tfrac{1}{2})h_{4n+1,0}$$
$$T_2 h_{4n+1,0} = 2((N-1)k+n)h_{4n-1,1}$$
$$T_2 h_{4n+3,1} = -2((N-1)k+k_1+n+\tfrac{3}{2})h_{4n+3,0}.$$

These formulae (together with the claim $T_i h_{n,\varepsilon} = 0$ for $i > 2$) already imply that $\Delta_B h_{n,\varepsilon} = 0$ because $T_1^2 h_{4n,0} = c h_{4n-2,0}$ for some constant c and $T_2^2 h_{4n,0} = \sigma_{12} T_1^2 \sigma_{12} h_{4n,0} = \sigma_{12} T_1^2 h_{4n,0} = -c h_{4n-2,0}$. All the other polynomials $h_{m,\varepsilon}$ can be obtained by applying T_1 or T_2 often enough to some $h_{4n,0}$ with $4n > m$. In subsequent sections we will derive the values of $\|h_{n,\varepsilon}\|_2^2$, $h_{n,\varepsilon}(1,1,\ldots,1)$, $h_{n,\varepsilon}(1,0,\ldots,0)$ and the coefficients of the leading terms. All but the L^2 norms are products of linear factors in the parameters, while the norms are expressed as sums of balanced ${}_4F_3$ series.

3. Symbolic calculus

The results described above depend on the basis of polynomials introduced in [5] for type A, [6] for type B. The idea is to replace the variables in the type-A basis by $x_1^2, x_2^2, \ldots, x_N^2$ and then use the expressions for T_i in terms of corresponding type-A operators. Throughout let $y = (y_1, y_2, \ldots, y_N) = (x_1^2, x_2^2, \ldots, x_N^2)$ for $x \in \mathbb{R}^N$. The type-A Dunkl operator is defined by

$$\hat{T}_i = \frac{\partial}{\partial y_i} + k \sum_{j \neq i} \frac{1-(ij)}{y_i - y_j}, \qquad 1 \leqslant i \leqslant N$$

where (ij) denotes the transposition of y_i and y_j, the effect of σ_{ij} or τ_{ij} on the squared variables. The polynomials in $x \in \mathbb{R}^N$ are spanned by polynomials of the form $x^\varepsilon g(y)$ where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ and $x^\varepsilon = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots$ with each $\varepsilon_i = 0$ or 1.

Proposition 1. *Let $f(x) = x^\varepsilon g(y)$ with each $\varepsilon_i = 0$ or 1. For $i = 1, 2, \ldots, N$,*

$$T_i f(x) = 2x_i x^\varepsilon \hat{T}_i g(y) \qquad \text{if} \quad \varepsilon_i = 0$$
$$T_i f(x) = 2\frac{x^\varepsilon}{x_i}\left(\left(k_1 - \frac{1}{2}\right)g + \hat{T}_i(y_i g) - k\sum_j \{(ij)g : \varepsilon_j = 1, j \neq i)\}\right) \qquad \text{if} \quad \varepsilon_i = 1.$$

This is proposition 2.1 in [6]. The p basis for the symmetric group action is constructed as follows: for $1 \leqslant i \leqslant N$ the polynomials $p_n(y_i; y)$ are given by the generating function

$$\sum_{n=0}^{\infty} p_n(y_i; y) r^n = (1 - r y_i)^{-1} \prod_{j=1}^{N} (1 - r y_j)^{-k}$$

then for $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$ the collection of compositions, the basis element $p_\alpha = \prod_{i=1}^N p_{\alpha_i}(y_i; y)$. The key property is that $\hat{T}_j p_n(y_i; y) = 0$ for $j \neq i$. It was shown in [5] that

$$\hat{T}_i p_\alpha = (Nk + \alpha_i) p_{\alpha_i - 1}(y_i; y) \prod_{m \neq i} p_{\alpha_m}(y_m; y)$$
$$+ k \sum_{j \neq i} \left\{ \sum_{m=0}^{\alpha_j - 1} (p_{\alpha_i + \alpha_j - 1 - m}(y_i; y) p_m(y_j; y) \right.$$
$$\left. - p_m(y_i; y) p_{\alpha_i + \alpha_j - 1 - m}(y_j; y)) \prod_{n \neq i, j} p_{\alpha_n}(y_n; y) \right\} \qquad (3.1)$$

if $\alpha_i > 0$, and $\hat{T}_i p_\alpha = 0$ if $\alpha_i = 0$. On the right-hand side of this formula the same term may appear twice, but this has no relevance for the intended use. We will set up a linear isomorphism between the span of the p basis and polynomials in the formal variables $p_1, p_2, \ldots, p_N$ which is not generally multiplicative but does allow a simple formula for $\hat{T}_i$.

Definition 2. *The linear isomorphism Ψ between the span of the p basis and the space $\mathcal{P}$ of polynomials in the formal variables $p_1, p_2, \ldots, p_N$ is given by $\Psi : p_\alpha \longmapsto p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_N^{\alpha_N}$ (and extended by linearity). Further, the linear transformations $\zeta_{i,j}$ and η_i on $\mathcal{P}$ are defined by $\zeta_{i,j} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_N^{\alpha_N} = p_i^{\alpha_i+\alpha_j} \prod_{m\neq i,j} p_m^{\alpha_m}$ for $i \neq j$, and η_i is evaluation at $p_i = 0$, for $1 \leqslant i \leqslant N$. (That is, $\zeta_{i,j}$ replaces p_j by p_i, and η_i replaces p_i by 0.)*

It is clear that Ψ commutes with the S_N action. We use the simplified notation $\hat{T}_i$ for the operator $\Psi \hat{T}_i \Psi^{-1}$ on $\mathcal{P}$.

Proposition 2. *For $1 \leqslant i \leqslant N$, the operator $\hat{T}_i$ acts on polynomials in $\mathcal{P}$ by*

$$\hat{T}_i = \frac{\partial}{\partial p_i} + Nk\frac{1-\eta_i}{p_i} + k\sum_{j\neq i} \frac{\zeta_{i,j} + \zeta_{j,i} - 1 - (i,j)}{p_i - p_j}.$$

Proof. It suffices to examine the effect of the formula on monomials $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_N^{\alpha_N}$ and for $i = 1$. The first two terms produce $(\alpha_1 + Nk)$ if $\alpha_1 > 0$, or otherwise 0. In the sum, the (typical) term for $j = 2$ is $(p_1^{\alpha_1+\alpha_2} + p_2^{\alpha_1+\alpha_2} - p_1^{\alpha_1} p_2^{\alpha_2} - p_1^{\alpha_2} p_2^{\alpha_1}) \prod_{m=3}^{N} p_m^{\alpha_m}/(p_1 - p_2)$. A simple calculation shows this is the image under Ψ of the corresponding term in equation (3.1). □

We will use generating functions to determine the effects of T_1 and T_2 on the polynomials defined in the previous sections (now we are considering the type-B operators). For this purpose we consider the cases $f(p), x_1 f(p), x_1x_2 f(p)$ where $f(p)$ is a formal series in $p = (p_1, p_2)$ (the validity of term-by-term action comes from the same argument used to justify term-by-term differentiation of a power series inside its disc of convergence). In the following, we use the notation

$$\delta_{1,2} f(p_1, p_2) = (f(p_1, p_1) + f(p_2, p_2) - f(p_1, p_2) - f(p_2, p_1))/(p_1 - p_2).$$

Proposition 3. *For a power series $f(p_1, p_2)$ (absolutely convergent in the region $\{(p_1, p_2) : |p_1| < 1, |p_2| < 1)$ the following hold:*

(1) $T_1 f(p) = 2x_1(\frac{\partial f(p)}{\partial p_1} + Nk\frac{f(p_1,p_2)-f(0,p_2)}{p_1} + k\delta_{1,2} f(p))$,

(2) $T_1(x_1 f(p)) = 2(((N-1)k + k_1 + \frac{1}{2}) f(p) + p_1\frac{\partial f(p)}{\partial p_1} + k\delta_{1,2}(p_1 f(p)))$,

(3) $T_2(x_1x_2 f(p)) = 2x_1(((N-1)k + k_1 + \frac{1}{2}) f(p) + p_2\frac{\partial f(p)}{\partial p_2} - k\delta_{1,2}(p_2 f(p)) - kf(p_2, p_1))$,

(4) $T_2(x_1 f(p)) = 2x_1x_2(\frac{\partial f(p)}{\partial p_2} + Nk\frac{f(p_1,p_2)-f(p_1,0)}{p_2} - k\delta_{1,2} f(p))$.

Proof. Formulae (1) and (4) follow immediately from proposition 1. It was shown in lemma 2.3 of [6] that $\hat{T}_i y_i = \hat{T}_i \hat{\rho}_i - k$, where $\hat{\rho}_i$ is the conjugate under Ψ of multiplication by p_i acting on $\mathcal{P}$. Together with proposition 2 this proves formulae (2) and (3). □

By the fundamental properties of the p basis, $T_i(x_1^{\varepsilon_1} x_2^{\varepsilon_2} f)(p_1, p_2) = 0$ for all $i > 2$, and $\varepsilon_1, \varepsilon_2 = 0$ or 1. This applies to all the polynomials used in what follows. The images under Ψ of the generating functions F_0, F_1 defined in section 2 are in fact simple rational functions in

$p = (p_1, p_2)$. Indeed, for indeterminates u_1, u_2, from the definition of the p basis it follows that

$$\begin{aligned}\Psi\left((1-u_1y_1)^{-1}(1-u_2y_2)^{-1}\prod_{i=1}^{N}\{(1-u_1y_i)(1-u_2y_i)\}^{-k}\right) &= \Psi\left(\sum_{m,n=0}^{\infty} p_{(m,n)}u_1^m u_2^n\right)\\ &= \sum_{m,n=0}^{\infty} p_1^m p_2^n u_1^m u_2^n = (1-u_1p_1)^{-1}(1-u_2p_2)^{-1}. \end{aligned}\tag{3.2}$$

The desired expressions result from changing variables to $u_1 = tz, u_2 = tz^{-1}$, and also $u_1 = tz^{-1}, u_2 = tz$ and $s = \frac{1}{2}(z + z^{-1})$; then the two terms are combined by addition and subtraction (symmetric and skew-symmetric under (1), (2), respectively). To ensure convergence some region must be chosen, for example, $|s| < \frac{4}{3}$ and $|t| < 1/(3\max(|p_1|, |p_2|))$ (and $s, t \in \mathbb{C}$). This is valid because $|r - \frac{1}{r}| \leqslant 2|s| \leqslant (r + \frac{1}{r})$ where $r = |z|$ and $z \in \mathbb{C}$, thus $|s| < \frac{4}{3}$ implies $\frac{1}{3} < |z| < 3$.

The method for computing the effect of T_i on the polynomials $\phi_{n,j}$ and $\psi_{n,j}$ is to apply T_i to the generating functions and express the result by means of combinations of $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ and multiplication by s, t. The two basic functions are

$$\begin{aligned} w_1 &= (1-ztp_1)^{-1}(1-z^{-1}tp_2)^{-1}\\ w_2 &= (1-z^{-1}tp_1)^{-1}(1-ztp_2)^{-1}. \end{aligned}$$

Then let

$$\begin{aligned} f_0 &= \tfrac{1}{2}(w_1+w_2) = \frac{1-st(p_1+p_2)+t^2p_1p_2}{(1-2stp_1+t^2p_1^2)(1-2stp_2+t^2p_2^2)}\\ f_1 &= (z-z^{-1})^{-1}(w_1-w_2) = \frac{t(p_1-p_2)}{(1-2stp_1+t^2p_1^2)(1-2stp_2+t^2p_2^2)}. \end{aligned}$$

The same formulae apply to the images under Ψ^{-1}.

Proposition 4. *The generating functions in definition 1 satisy* $\Psi F_0 = f_0$ *and* $\Psi F_1 = f_1$.

Proof. Apply Ψ^{-1} to w_1 and w_2 using equation (3.2), then both $\Psi^{-1}w_1$ and $\Psi^{-1}w_2$ have the common factor $\prod_{i=1}^{N}((1-u_1y_i)(1-u_2y_i))^{-k} = \prod_{i=1}^{N}(1-2sty_i+t^2y_i^2)^{-k}$. The parts of the calculation involving $(1-u_iy_1)^{-1}$ and $(1-u_iy_2)^{-1}$, $i = 1$ or 2 proceed just as those with w_1 and w_2. □

3.1. Action of T_i on the generating functions

Now we can use the symbolic calculus on f_0 and f_1. Write $g_0 = f_0 + sf_1$ and $g_1 = -f_1$ for the generating functions for $\{\psi_{n,j}\}$. Then $w_1 = f_0 + \frac{1}{2}(z - z^{-1})f_1 = g_0 + z^{-1}g_1$ and $w_2 = f_0 - \frac{1}{2}(z - z^{-1})f_1 = g_0 + zg_1$. First, the effect of $\delta_{1,2}$ on various functions is calculated: $\delta_{1,2}f_0 = tf_1, \delta_{1,2}f_1 = 0, \delta_{1,2}(p_2f_0) = sf_1, \delta_{1,2}(p_2f_1) = f_1, \delta_{1,2}(g_0) = tf_1, \delta_{1,2}(g_1) = 0, \delta_{1,2}(p_1g_0) = 0, \delta_{1,2}(p_1g_1) = f_1$. These simple relations are the reason for using this particular set of functions.

The differentiations can be done on w_1 and w_2, separately. It is easy to verify that

$$\frac{\partial w_1}{\partial p_1} = zt\left(w_1 + \frac{t}{2}\frac{\partial w_1}{\partial t}\right) + \frac{z^2 t}{2}\frac{\partial w_1}{\partial z}$$
$$\frac{\partial w_2}{\partial p_1} = \frac{t}{z}\left(w_2 + \frac{t}{2}\frac{\partial w_2}{\partial t}\right) - \frac{t}{2}\frac{\partial w_2}{\partial z}.$$

Further, $p_1\frac{\partial w_1}{\partial p_1} = \frac{1}{2}(\frac{\partial}{\partial t} - \frac{\partial}{\partial z})w_1$, $p_2\frac{\partial w_1}{\partial p_1} = \frac{1}{2}(\frac{\partial}{\partial t} + \frac{\partial}{\partial z})w_2$, $(1-\eta_1)w_1/p_1 = ztw_1$ and $(1-\eta_1)w_2/p_1 = (t/z)w_2$. In the expressions for $\frac{\partial f_0}{\partial p_1}$ and $\frac{\partial f_1}{\partial p_1}$ the following equations are used:

$$zw_1 + z^{-1}w_2 = 2sg_0 + 2g_1 \qquad z^2\frac{\partial w_1}{\partial z} - \frac{\partial w_2}{\partial z} = 2(s^2-1)\frac{\partial g_0}{\partial s} - 2g_1$$
$$(z-z^{-1})^{-1}(zw_1 - z^{-1}w_2) = g_0 \qquad (z-z^{-1})^{-1}\left(z^2\frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial z}\right) = s\frac{\partial g_0}{\partial s} + \frac{\partial g_1}{\partial s}$$
$$z\left(\frac{\partial w_1}{\partial z} - \frac{\partial w_2}{\partial z}\right) = -2sg_1 + (1-s^2)\frac{\partial g_1}{\partial s}$$
$$z(z-z^{-1})^{-1}\left(\frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial z}\right) = \frac{\partial g_0}{\partial s} + g_1 + s\frac{\partial g_1}{\partial s}.$$

These, as well as the following equations can be proven by direct verification (express w_1 and w_2 in terms of g_0 and g_1, or f_0 and f_1; of course $\frac{\partial}{\partial z} = \frac{1}{2}(1-z^{-2})\frac{\partial}{\partial s}$). The formulae are grouped by type as given in sections 3.1.1–3.1.4.

3.1.1. Case $T_1 f(y) : \phi \to \psi$.

$$T_1 f_0 = 2x_1 t\left(\left[(Nk+1)s + \frac{st}{2}\frac{\partial}{\partial t} + \frac{s^2-1}{2}\frac{\partial}{\partial s}\right]g_0 + \left[(N-1)k + \frac{1}{2} + \frac{t}{2}\frac{\partial}{\partial t}\right]g_1\right)$$
$$T_1 f_1 = 2x_1 t\left(\left[(Nk+1) + \frac{t}{2}\frac{\partial}{\partial t} + \frac{s}{2}\frac{\partial}{\partial s}\right]g_0 + \frac{1}{2}\frac{\partial}{\partial s}g_1\right).$$

3.1.2. Case $T_2(x_1x_2 f(y)) : x_1x_2\phi \to \psi$.

$$T_2(x_1x_2f_0) = 2x_1\left(\left[(N-2)k + k_1 + \frac{1}{2} + \frac{t}{2}\frac{\partial}{\partial t}\right]g_0\right.$$
$$\left. + \left[s\left((N-1)k + k_1 + 1 + \frac{t}{2}\frac{\partial}{\partial t}\right) + \frac{s^2-1}{2}\frac{\partial}{\partial s}\right]g_1\right)$$
$$T_2(x_1x_2f_1) = 2x_1\left(-\frac{1}{2}\frac{\partial}{\partial s}g_0 - \left[(N-1)k + k_1 + 1 + \frac{t}{2}\frac{\partial}{\partial t} + \frac{s}{2}\frac{\partial}{\partial s}\right]g_1\right).$$

3.1.3. Case $T_1(x_1 f(y)) : \psi \to \phi$.

$$T_1(x_1g_0) = 2\left(\left[(N-1)k + k_1 + \frac{1}{2} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{s}{2}\frac{\partial}{\partial s}\right]f_0\right.$$
$$\left. + \left[s\left((N-1)k + k_1 + 1 + \frac{t}{2}\frac{\partial}{\partial t}\right) + \frac{s^2-1}{2}\frac{\partial}{\partial s}\right]f_1\right)$$
$$T_1(x_1g_1) = 2\left(-\frac{1}{2}\frac{\partial}{\partial s}f_0 - \left[(N-2)k + k_1 + \frac{1}{2} + \frac{t}{2}\frac{\partial}{\partial t}\right]f_1\right).$$

3.1.4. Case $T_2(x_1 f(y)) : \psi \to x_1 x_2 \phi$.

$$T_2(x_1 g_0) = 2x_1x_2t\left(-\frac{1}{2}\frac{\partial}{\partial s}f_0 + \left[(N-1)k + 1 + \frac{t}{2}\frac{\partial}{\partial t}\right]f_1\right)$$

$$T_2(x_1 g_1) = 2x_1x_2t\left(\left[(Nk+1) + \frac{t}{2}\frac{\partial}{\partial t} + \frac{s}{2}\frac{\partial}{\partial s}\right]f_0 - \left[s\left(Nk + \frac{3}{2} + \frac{t}{2}\frac{\partial}{\partial t}\right) + \frac{1}{2}(s^2-1)\frac{\partial}{\partial s}\right]f_1\right).$$

3.2. Action of T_i on basis polynomials

Perusal of these formulae reveals that terms involving k are almost 'on the diagonal', that is, setting $k = 0$ does not noticeably simplify the formulae. In the $\{\phi_{n,j}, \psi_{n,j}\}$ basis, the $k \neq 0$ case is no more complicated than $k = 0$. This illustrates the advantage of these polynomials over the ordinary x basis. In each of the formulae, the result of expanding the equations in $\{\phi_{n,j}\}$ for f_i and in $\{\psi_{n,j}\}$ for g_i, ($i = 0$ or 1) and matching up coefficients of $s^j t^n$ on both sides leads to the following (grouped by the parity of $n + j$) results.

3.2.1. $n + j = 0 \bmod 2$.

$$T_1\phi_{n,j} = (2Nk + n + j)\psi_{n-1,j-1} + (2(N-1)k + n)\psi_{n-1,j} - (j+1)\psi_{n-1,j+1}$$

$$T_1\psi_{n,j} = (2(N-1)k + 2k_1 + n + j + 1)(\phi_{n,j} + \phi_{n,j-1}) - (j+1)\phi_{n,j+1}$$

$$T_2(x_1x_2\phi_{n,j}) = (2(N-2)k + 2k_1 + n + 1)\psi_{n,j} + (2(N-1)k + 2k_1 + n + j + 1)\psi_{n,j-1} - (j+1)\psi_{n,j+1}$$

$$T_2\psi_{n,j} = x_1x_2((2(N-1)k + n + 1)\phi_{n-1,j} - (j+1)\phi_{n-1,j+1}).$$

3.2.2. $n + j = 1 \bmod 2$.

$$T_1\phi_{n,j} = (2Nk + n + j + 1)\psi_{n-1,j} + (j+1)\psi_{n-1,j+1}$$

$$T_1\psi_{n,j} = -(2(N-2)k + 2k_1 + n + 1)\phi_{n,j} - (j+1)\phi_{n,j+1}$$

$$T_2(x_1x_2\phi_{n,j}) = -(2(N-1)k + 2k_1 + n + j + 2)\psi_{n,j} - (j+1)\psi_{n,j+1}$$

$$T_2\psi_{n,j} = x_1x_2((2Nk + n + j + 1)(\phi_{n-1,j} - \phi_{n,j-1}) + (j+1)\phi_{n-1,j+1}).$$

Notice that each expression has no more than three different polynomials on the right-hand side. In the next section we use these to determine the harmonic polynomials.

4. Properties of the harmonic polynomials

Here we demonstrate the action of T_1, T_2 on the harmonic polynomials, which suffices to show $\Delta_B h_{n,\varepsilon} = 0$ for each such polynomial, as mentioned before. By construction the polynomials defined in section 2.1 satisfy $T_i h_{n,\varepsilon} = 0$ for $i > 2$; which is sufficient to establish the formulae of section 2.2. In a sense the proofs depend on induction. Since the computations of $T_i h_{n,\varepsilon}$ are somewhat repetitive we will not give details on each formula. The calculations are direct; the definitions of F_0, F_1 and $h_{n,\varepsilon}$ were formulated after computer-algebra-aided experimentation. In addition, we determine the values at $(1, 1, \ldots, 1)$, $(1, 0, \ldots, 0)$ and the L^2 norms.

4.1. The action of T_i on h

There are four main cases, each with two parts. Because k_1 always appears in the same way we introduce the (abbreviation) notation:

$$k_2 = (N-1)k + k_1 + \tfrac{1}{2}.$$

4.1.1. Case $T_1 : h_{2m,0} \to h_{2m-1,0}$. We begin with the proof of $T_1 h_{4n,0} = 2((N-1)k + n)h_{4n-1,0}$, for $n \geqslant 1$. We write $T_1 \sum_{i=0}^{m} c_i \phi_{m,i} = \sum_{i=0}^{m-1} (T_1^* c)_i \psi_{m-1,i}$, then

$$(T_1^*c)_i = (2(N-1)k + m)c_i + ic_{i-1} \qquad \text{for} \quad m+i = 0 \bmod 2$$
$$(T_1^*c)_i = (2Nk + m + i + 1)(c_{i+1} + c_i) - ic_{i-1} \qquad \text{for} \quad m+i = 1 \bmod 2.$$

These follow from the equations in section 3.2. Now let $h_{4n,0} = \sum_{j=0}^{n} a_j \phi_{2n,2j}$ with $a_j = \frac{((N-1)k+k_2+2n)_j (\frac{1}{2})_j}{(k_2+n)_j (Nk+n+1)_j}$. Use the above equations with $m = 2n$, then $T_1 h_{4n,0} = \sum_{j=0}^{n-1} b_j \psi_{2n-1,2j} + \sum_{j=1}^{n} c_j \psi_{2n-1,2j-1}$ with $b_j = 2((N-1)k + n)a_j$ and

$$\begin{aligned} c_j &= 2(Nk + n + j)a_j - 2(j - \tfrac{1}{2})a_{j-1} \\ &= 2((N-1)k + n) \frac{((N-1)k + k_2 + 2n)_{j-1} (\frac{1}{2})_j}{(k_2 + n)_j (Nk + n + 1)_{j-1}} \end{aligned}$$

which is the claimed result. Similarly, one can show $T_1 h_{4n+2,0} = 2(Nk + n + 1)h_{4n+1,0}$.

4.1.2. Case $T_1 : h_{2m+1,0} \to h_{2m,0}$. Next we consider the case $T_1 \sum_{i=0}^{m} c_i \psi_{m,i} = \sum_{i=0}^{m} (T_1^* c)_i \phi_{m,i}$, where (see section 3.2)

$$(T_1^*c)_i = (2k_2 + m + i)c_i - ic_{i-1} \qquad \text{for} \quad m+i = 0 \bmod 2$$
$$(T_1^*c)_i = (2k_2 + m + i + 1)c_{i+1} - (2k_2 - 2k + m)c_i - ic_{i-1} \qquad \text{for} \quad m+i = 1 \bmod 2.$$

Write

$$h_{4n+3,0} = \sum_{j=0}^{n} b_j \psi_{2n+1,2j} + \sum_{j=1}^{n+1} c_j \psi_{2n+1,2j-1}$$

then

$$T_1 h_{4n+3,0} = \sum_{j=0}^{n} a_j \phi_{2n+1,2j} + \sum_{j=1}^{n+1} d_j \phi_{2n+1,2j-1}$$

with $d_j = 2(k_2 + n + j)c_j - (2j-1)b_{j-1} = 0$ and

$$\begin{aligned} a_j &= 2(k_2 + n + j + 1)c_{j+1} - (2k_2 - 2k + 2n + 1)b_j - 2jc_j \\ &= -2\left((N-2)k + k_1 + n + \frac{1}{2}\right) \frac{((N-1)k + k_2 + 2n + 1)_j (\frac{1}{2})_j}{(k_2 + n + 1)_j (Nk + n + 2)_j} \end{aligned}$$

the claimed multiple of $h_{4n+2,0}$. Similarly,

$$h_{4n+1,0} = \sum_{j=0}^{n} b_j \psi_{2n,2j} + \sum_{j=1}^{n} c_j \psi_{2n,2j-1}$$
$$T_1 h_{4n+1,0} = \sum_{j=0}^{n} a_j \phi_{2n,2j} + \sum_{j=1}^{n} d_j \phi_{2n,2j-1}$$

with $a_j = 2(k_2 + n + j)b_j - 2jc_j$ which is $2(k_2 + n)$ times the corresponding coefficient of $h_{4n,0}$, while $d_j = 2(k_2 + n + j)b_j - 2(k_2 - k + n)c_j - (2j-1)b_{j-1} = 0$.

4.1.3. Case $T_2 : h_{2m+1,0} \to h_{2m-1,1}$. For $T_2 \sum_{i=0}^{m} c_i \psi_{m,i} = \sum_{i=0}^{m-1} (T_2^* c)_i (x_1 x_2 \phi_{m-1,i})$ one has

$$(T_2^* c)_i = (2(N-1)k + m + 1)c_i - (2Nk + m + i + 2)c_{i+1} + i c_{i-1} \qquad \text{for} \quad m + i = 0 \bmod 2$$
$$(T_2^* c)_i = (2Nk + m + i + 1)c_i - i c_{i-1} \qquad \text{for} \quad m + i = 1 \bmod 2.$$

4.1.4. Case $T_2 : h_{2m+1,1} \to h_{2m+1,0}$. For $T_2 \sum_{i=0}^{m} c_i x_1 x_2 \phi_{m,i} = \sum_{i=0}^{m} (T_2^* c)_i \psi_{m,i}$ one has

$$(T_2^* c)_i = (2k_2 - 2k + m)c_i - i c_{i-1} \qquad \text{for} \quad m + i = 0 \bmod 2$$
$$(T_2^* c)_i = (2k_2 + m + i + 1)(c_{i+1} - c_i) - i c_{i-1} \qquad \text{for} \quad m + i = 1 \bmod 2.$$

4.2. Values at $(1, 1, \ldots, 1)$

Substituting $x = 1^N = (1, 1, \ldots, 1) \in \mathbb{R}^N$ in F_0 and F_1 produces

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \phi_{n,j}(1^N) s^j t^n = (1 - 2st + t^2)^{-(Nk+1)}$$
$$= \sum_{m,i} t^m s^{m-2i} 2^{m-2i} (-1)^i \frac{(Nk+1)_{m-i}}{i!(m-2i)!}$$

since only terms with $n + j = 0 \bmod 2$ can have nonzero values; note that the second equality is familiar as the generating function for Gegenbauer polynomials. To derive this expansion, write

$$(1 - 2st + t^2)^{-(Nk+1)} = (1 + t^2)^{-(Nk+1)} \left(1 - \frac{2st}{1 + t^2}\right)^{-(Nk+1)}$$
$$= \sum_{j=0}^{\infty} \frac{(Nk+1)_j}{j!} (2st)^j (1 + t^2)^{-(Nk+1+j)}$$
$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(Nk+1)_{j+i}}{j!i!} (2s)^j (-1)^i t^{2i+j}$$

now let $j = m - 2i$.

Proposition 5. *For* $n = 0, 1, 2, \ldots$ *the following hold:*

$$h_{4n,0}(1^N) = \frac{(Nk+1)_n((N-1)k+1)_n}{n!(k_2+n)_n}$$
$$h_{4n+1,0}(1^N) = \frac{(Nk+1)_n((N-1)k+1)_n}{n!(k_2+n+1)_n}$$
$$h_{4n+2,0}(1^N) = h_{4n+3,1}(1^N) = 0$$
$$h_{4n+1,1}(1^N) = \frac{(Nk+1)_n((N-1)k+1)_n}{n!(k_2+n+1)_n}$$
$$h_{4n+3,0}(1^N) = \frac{(Nk+1)_{n+1}((N-1)k+1)_n}{n!(k_2+n+1)_{n+1}}.$$

Proof. The nonzero cases are all ${}_2F_1$ summations. For the first case, $\phi_{2n,2j}(1^N) = \frac{2^{2j}(Nk+1)_{n+j}(-1)^{n-j}}{(n-j)!(2j)!} = \frac{(-1)^n(-n)_j(Nk+1)_{n+j}}{n!j!(\frac{1}{2})_j}$, thus

$$h_{4n,0}(1^N) = \frac{(-1)^n(Nk+1)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j(k_2 + (N-1)k + 2n)_j}{(k_2+n)_j\, j!}$$

$$= \frac{(-1)^n (Nk+1)_n (-(N-1)k-n)_n}{n!(k_2+n)_n}.$$

This uses the Chu–Vandermonde sum ${}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix}; 1\right) = \frac{(c-b)_n}{(c)_n}$; and $(-b-n)_n = (-1)^n(b+1)_n$ for arbitrary b, c and $n = 0, 1, 2, \ldots$. The other formulae are proved in the same way. □

4.3. Leading coefficients

Let $\mathrm{cof}(f, x_1^m x_2^n)$ denote the coefficient of the monomial $x_1^m x_2^n$ in the expansion of the polynomial f in terms of $x_1, x_2, \ldots, x_N$. We will determine the values of $\mathrm{cof}(h_{m,\varepsilon}, x_1^m x_2^\varepsilon)$. For the case $\varepsilon = 0$ these values agree with evaluation at $(1, 0, \ldots, 0)$. Thus evaluate F_0 and F_1 at this point to obtain $(1-st)(1-2st+t^2)^{-(k+1)} = a_0$ and $t(1-2st+t^2)^{-(k+1)} = a_1$, respectively. The term a_1 is multiplied by (-1) to obtain coefficients of x_2^n in $h_{n,0}$. By expansion methods similar to those used previously we obtain

$$a_0 = \sum_{m,i} t^m s^{m-2i} 2^{m-2i-1} (-1)^i \frac{(k+1)_{m-1-i}(2k+m)}{i!(m-2i)!}$$

$$a_1 = \sum_{m,i} t^{m+1} s^{m-2i} 2^{m-2i} (-1)^i \frac{(k+1)_{m-i}}{i!(m-2i)!}.$$

4.3.1. Case $(4n+\varepsilon, \varepsilon)$. The computation for $h_{4n,0}(1, 0, \ldots)$ proceeds as follows:

$$\phi_{2n,2j}(1, 0, \ldots) = (-1)^n \frac{(-n)_j (k+1)_n (k+n)_j}{n!\, j!\, (\frac{1}{2})_j}$$

and so

$$h_{4n,0}(1, 0, \ldots) = \frac{(-1)^n (k+1)_n}{n!} {}_3F_2\left(\begin{matrix}-n, k+n, k_2+(N-1)k+2n\\ k_2+n, Nk+n+1\end{matrix}; 1\right)$$
$$= \frac{(-1)^n (k+1)_n (k_2-k)_n ((N-1)k+1)_n}{n!(k_2+n)_n (Nk+n+1)_n}.$$

Clearly, $\mathrm{cof}(h_{4n,0}, x_1^{4n}) = \mathrm{cof}(h_{4n,0}, x_2^{4n}) = h_{4n,0}(1, 0, \ldots)$. The sum is an application of the Saalschütz formula ${}_3F_2(\begin{matrix}-n, a, b\\ c, d\end{matrix}; 1) = \frac{(c-a)_n(d-a)_n}{(c)_n(d)_n}$, provided $-n+a+b+1 = c+d$. The corresponding formulae for $h_{4n+1,1}$ are obtained by merely incrementing k_1 (and also k_2) by 1. Thus

$$\mathrm{cof}(h_{4n+1,1}, x_1^{4n+1}x_2) = \mathrm{cof}(h_{4n+1,1}, x_1 x_2^{4n+1})$$
$$= \frac{(-1)^n (k+1)_n (k_2+1-k)_n ((N-1)k+1)_n}{n!(k_2+n+1)_n (Nk+n+1)_n}.$$

4.3.2. Case $(4n+2+\varepsilon, \varepsilon)$. For $h_{4n+2,0}(1, 0, \ldots)$ we begin with $\phi_{2n+1,2j}(1, 0, \ldots) = (-1)^n \frac{(-n)_j (k+1)_{n+j}}{n!\, j!\, (\frac{1}{2})_j}$ and

$$h_{4n+2,0}(1, 0, \ldots) = \frac{(-1)^n (k+1)_n}{n!} {}_3F_2\left(\begin{matrix}-n, k+n+1, k_2+(N-1)k+2n+1\\ k_2+n+1, Nk+n+2\end{matrix}; 1\right)$$
$$= \frac{(-1)^n (k+1)_n (k_2-k)_n ((N-1)k+1)_n}{n!(k_2+n+1)_n (Nk+n+2)_n}.$$

Further, $\mathrm{cof}(h_{4n+2,0}, x_1^{4n+2}) = -\mathrm{cof}(h_{4n+2,0}, x_2^{4n+2}) = h_{4n+2,0}(1, 0, \ldots)$. Replace k_2 by k_2+1 to obtain the value of $\mathrm{cof}(h_{4n+3,1}, x_1^{4n+3}x_2) = -\mathrm{cof}(h_{4n+3,1}, x_1 x_2^{4n+3})$.

4.3.3. Case $(4n+1,0)$. For $h_{4n+1,0}$ note that $\psi_{2n,2j} = x_1(\phi_{2n,2j} + \phi_{2n,2j-1})$ and $\psi_{2n,2j-1} = -x_1\phi_{2n,2j-1}$. To sketch the argument, let $h_{4n+1,0} = \sum_{j=0}^{n} a_j\psi_{2n,2j} + \sum_{j=1}^{n} b_j\psi_{2n,2j-1}$ and let $\phi_{2n,2j}(1,0,\ldots) = c_j$ and $\phi_{2n,2j-1}(1,0,\ldots) = d_j$. Then $\sum_{j=0}^{n} a_jc_j + \varepsilon\sum_{j=1}^{n}(a_j - b_j)d_j$ equals $cof(h_{4n+1,0}, x_1^{4n+1})$ when $\varepsilon = 1$, and $\operatorname{cof}(h_{4n+1,0}, x_1x_2^{4n})$ when $\varepsilon = -1$. The value of c_j was found above, and thus the first sum equals $\frac{(-1)^n(k+1)_n(k_2-k+1)_n((N-1)k+1)_n}{n!(k_2+n+1)_n(Nk+n+1)_n}$. For the second sum,

$$a_j - b_j = ((N-1)k+n)\frac{(k_2+(N-1)k+2n+1)_{j-1}(\frac{1}{2})_j}{(k_2+n+1)_j(Nk+n+1)_j}.$$

The value of d_j is calculated similarly to $\phi_{2n+1,2j}$ and thus the second sum equals

$$\frac{((N-1)k+n)(-1)^{n-1}(k+1)_n}{(n-1)!(k_2+n+1)(Nk+n+1)}{}_3F_2\left(\begin{matrix}1-n, k+n+1, k_2+(N-1)k+2n+1\\ k_2+n+2, Nk+n+2\end{matrix};1\right)$$
$$= \frac{(-1)^{n-1}(k+1)_n(k_2-k+1)_{n-1}((N-1)k+1)_n}{(n-1)!(k_2+n+1)_n(Nk+n+1)_n}.$$

Combining the two sums and setting $\varepsilon = 1$ and -1, respectively, we obtain

$$\operatorname{cof}(h_{4n+1,0}, x_1^{4n+1}) = \frac{(-1)^n(k+1)_n(k_2-k)_n((N-1)k+1)_n}{n!(k_2+n+1)_n(Nk+n+1)_n}$$

and

$$\operatorname{cof}(h_{4n+1,0}, x_1x_2^{4n}) = \operatorname{cof}(h_{4n+1,0}, x_1^{4n+1})\frac{k_2-k+2n}{k_2-k}.$$

4.3.4. Case $(4n+3,0)$. For $h_{4n+3,0}$ note that $\psi_{2n+1,2j} = -x_1\phi_{2n+1,2j}$ and $\psi_{2n+1,2j-1} = x_1(\phi_{2n+1,2j-1} + \phi_{2n+1,2j-2})$. As before, let $h_{4n+3,0} = \sum_{j=0}^{n} a_j\psi_{2n+1,2j} + \sum_{j=1}^{n+1} b_j\psi_{2n+1,2j-1}$ (not the same coefficients as above) and let $\phi_{2n+1,2j-1}(1,0,\ldots) = c_j$ and $\phi_{2n+1,2j}(1,0,\ldots) = d_j$. Then $\varepsilon\sum_{j=0}^{n}(-a_j + b_{j+1})d_j + \sum_{j=1}^{n+1} b_jc_j$ equals $\operatorname{cof}(h_{4n+3,0}, x_1^{4n+3})$ when $\varepsilon = 1$ and $\operatorname{cof}(h_{4n+3,0}, x_1x_2^{4n+2})$ when $\varepsilon = -1$. By a calculation similar to the previous one the first sum is found to equal

$$\left(k_2+n+\frac{1}{2}\right)\frac{(-1)^{n+1}(k+1)_n(k_2-k+1)_n((N-1)k+1)_n}{n!(k_2+n+1)_{n+1}(Nk+n+2)_n}$$

and the second sum is

$$\left(k+n+\frac{1}{2}\right)\frac{(-1)^n(k+1)_n(k_2-k+1)_n((N-1)k+1)_n}{n!(k_2+n+1)_{n+1}(Nk+n+2)_n}.$$

Combining the two sums and setting $\varepsilon = 1$ and -1, respectively, we obtain

$$\operatorname{cof}(h_{4n+3,0}, x_1^{4n+3}) = \frac{(-1)^{n+1}(k+1)_n(k_2-k)_{n+1}((N-1)k+1)_n}{n!(k_2+n+1)_{n+1}(Nk+n+2)_n}$$

and $\operatorname{cof}(h_{4n+3,0}, x_1x_2^{4n+2}) = -\operatorname{cof}(h_{4n+3,0}, x_1^{4n+3})\frac{k_2+k+2n+1}{k_2-k}$.

4.4. Norms

For arbitrary polynomials three different inner products have been defined. However, for harmonic polynomials there is really only one. Write

$$\mathrm{d}\mu_S(x;k,k_1) = \prod_{i=1}^{N}|x_i|^{2k_1}\prod_{1\leqslant i<j\leqslant N}|x_i^2 - x_j^2|^{2k}\,\mathrm{d}\omega(x)$$

for the measure on the unit sphere $S = \{x \in \mathbb{R}^N : |x| = 1\}$, where $\mathrm{d}\omega$ denotes the normalized roation-invariant surface measure. See formula (2.1) for the definition of $\mathrm{d}\mu(x;k,k_1)$.

Proposition 6. *Suppose f, g are harmonic ($\Delta_B f = 0 = \Delta_B g$) homogeneous polynomials of degree m and n, respectively. Then*

$$f(T_1, T_2, \ldots)g(x)|_{x=0} = c \int_{\mathbb{R}^N} f(x)g(x)\,\mathrm{d}\mu\,(x; k, k_1)$$
$$= \delta_{mn} 2^n (Nk_2)_n c_S \int_S fg\,\mathrm{d}\mu_S.$$

This was shown in theorem 3.8 of [3]; the normalizing constants satisfy $c\int_{\mathbb{R}^N}\mathrm{d}\mu = 1 = c_S \int_S \mathrm{d}\mu_S$, (the well known Macdonald–Mehta–Selberg integral). Specializing to the harmonic polynomials $g = h_{n,\varepsilon}$, for which $T_2^2 g = -T_1^2 g$ we see that if $f(x) = x_1^{\varepsilon_1} x_2^{\varepsilon_2} f_0(x_1^2, x_2^2, \ldots, x_N^2)$ with $\varepsilon_1, \varepsilon_2 = 0$ or 1, and f_0 is homogeneous of degree m then

$$f(T)g(x) = T_1^{\varepsilon_1} T_2^{\varepsilon_2} f_0(T_1^2, -T_1^2, 0, \ldots, 0)g(x) = f_0(1, -1, 0, \ldots, 0) T_1^{\varepsilon_1 + 2m} T_2^{\varepsilon_2} g(x).$$

By construction the polynomials $h_{n,\varepsilon}$ are pairwise orthogonal so only $h_{n,\varepsilon}(T)h_{n,\varepsilon}(x)$ need be computed. We begin with the calculation of $T_1^n T_2^\varepsilon h_{n,\varepsilon}$. The answers are best stated using a notation introduced in [6] as given in the following definition.

Definition 3. *For $m, n \in \mathbb{Z}_+$ and $m \geqslant n$ let*

$$\Lambda(m, n) = (Nk + 1)_a((N - 1)k + 1)_b (k_2)_{m-a} (k_2 - k)_{n-b}$$

where $a = \lfloor \frac{m}{2} \rfloor$ and $b = \lfloor \frac{n}{2} \rfloor$.

This is a special case of the generalized Pochhammer symbol for two-part partitions. From the formulae in section 2.2 we have

$$\begin{aligned}
&T_1^{4n} h_{4n,0} = 2^{4n}(-1)^n \Lambda(2n, 2n) \\
&T_1^{4n+1} h_{4n+1,0} = 2^{4n+1}(-1)^n \Lambda(2n + 1, 2n) \\
&T_1^{4n+2} h_{4n+2,0} = 2^{4n+2}(-1)^n \Lambda(2n + 2, 2n) \\
&T_1^{4n+1} T_2 h_{4n+1,1} = 2^{4n+2}(-1)^n \Lambda(2n + 1, 2n + 1) \\
&T_1^{4n+3} h_{4n+3,0} = 2^{4n+3}(-1)^{n+1} \Lambda(2n + 2, 2n + 1) \\
&T_1^{4n+3} T_2 h_{4n+3,1} = 2^{4n+4}(-1)^n \Lambda(2n + 3, 2n + 1).
\end{aligned}$$

We turn to the problem of the evaluations at $(1, -1, 0, \ldots, 0)$. In each case, the value will be expressed in terms of a balanced ${}_4F_3$ series which is obviously positive. This is the result of applying the Whipple transformation:

$${}_4F_3\left(\begin{matrix} -n, a, b, c \\ d, e, f \end{matrix}; 1\right) = \frac{(1 + a - e - n)_n (1 + a - f - n)_n}{(e)_n (f)_n}$$
$$\times {}_4F_3\left(\begin{matrix} -n, a, d - b, d - c \\ d, 1 + a - e - n, 1 + a - f - n \end{matrix}; 1\right)$$

provided $-n+a+b+c+1 = d+e+f$ (balanced), and $n \in \mathbb{Z}_+$. Setting $x = x_0 = (1, \sqrt{-1}, 0, \ldots)$ in the basis polynomials produces the desired values.

Lemma 1. *For $0 \leqslant j \leqslant n$, $\phi_{2n,2j+1}(x_0) = 0$, $\phi_{2n+1,2j+1}(x_0) = 0$ and*

$$\phi_{2n,2j}(x_0) = \frac{(2k + 1)_{n+j}(-1)^n(-n)_j(2k + 2n + 1)}{n!\,j!(k + \frac{3}{2})_j(2k + 1)}$$
$$\phi_{2n+1,2j}(x_0) = \frac{2(2k + 2)_{n+j}(-1)^n(-n)_j}{n!\,j!(k + \frac{3}{2})_j}.$$

Proof. Substituting $x = x_0$ in the generating functions yields $\sum_{n=0}^{\infty}\sum_{j=0}^{n}\phi_{n,j}(x_0) = (1+2t-t^2)((1+t^2)^2-4s^2t^2)^{-(k+1)}$. The latter term expands to

$$\sum_{i,j=0}^{\infty} s^{2j}t^{2i+2j}(-1)^i \frac{2^{2j}(k+1)_j(2k+2+2j)_i}{j!i!}.$$

Now multiply top and bottom by $(2k+2)_{2j}$, replace $i+j$ by n, (also a simple calculation to multiply the resulting series by $(1-t^2)$), and in the denominator expand $(2k+2)_{2j} = 2^{2j}(k+1)_j(k+\frac{3}{2})_j$. □

To illustrate the intermediate steps, consider the case

$$h_{4n,0}(x_0) = (-1)^n \frac{(2k+1)_n(2k+2n+1)}{n!(2k+1)} {}_4F_3\left(\begin{matrix} -n, k_2+(N-1)k+2n, \frac{1}{2}, 2k+n+1 \\ k+\frac{3}{2}, k_2+n, Nk+n+1 \end{matrix}; 1\right)$$

and transform the series, using $a = k_2+(N-1)k+2n$ and $d = k+\frac{3}{2}$. The other cases are done similarly (the case $h_{4n+3,0}$ incorporates one additional step, see section 4.3.4 above). The results are:

$$h_{4n,0}(x_0) = (-1)^n \frac{(2k+1)_n(2k+2n+1)((N-1)k+1)_n(k_2-k)_n}{n!(2k+1)(k_2+n)_n(Nk+n+1)_n} \times {}_4F_3\left(\begin{matrix} -n, k_2+(N-1)k+2n, k+1, -n-k+\frac{1}{2} \\ k+\frac{3}{2}, k_2-k, (N-1)k+1 \end{matrix}; 1\right)$$

$$h_{4n+2,0}(x_0) = (-1)^n \frac{2(2k+2)_n((N-1)k+1)_n(k_2-k)_n}{n!(k_2+n+1)_n(Nk+n+2)_n} \times {}_4F_3\left(\begin{matrix} -n, k_2+(N-1)k+2n+1, k+1, -n-k-\frac{1}{2} \\ k+\frac{3}{2}, k_2-k, (N-1)k+1 \end{matrix}; 1\right)$$

$$h_{4n+1,0}(x_0) = (-1)^n \frac{(2k+1)_n(2k+2n+1)((N-1)k+1)_n(k_2-k+1)_n}{n!(2k+1)(k_2+n+1)_n(Nk+n+1)_n} \times {}_4F_3\left(\begin{matrix} -n, k_2+(N-1)k+2n+1, k+1, -n-k+\frac{1}{2} \\ k+\frac{3}{2}, k_2-k+1, (N-1)k+1 \end{matrix}; 1\right)$$

$$h_{4n+3,0}(x_0) = (-1)^{n+1} \frac{(2k_2+2n+1)(2k+2)_n((N-1)k+1)_n(k_2-k+1)_n}{n!(k_2+n+1)_{n+1}(Nk+n+2)_n} \times {}_4F_3\left(\begin{matrix} -n, k_2+(N-1)k+2n+2, k+1, -n-k-\frac{1}{2} \\ k+\frac{3}{2}, k_2-k+1, (N-1)k+1 \end{matrix}; 1\right).$$

The values of the even parts of $h_{4n+1,1}(x_0)$ and $h_{4n+3,1}(x_0)$ are obtained by replacing k_2 by k_2+1 in $h_{4n+0,0}(x_0)$ and $h_{4n+2,0}(x_0)$, respectively ('even part' refers to f_0 in the expressions $h_{2m+1,1}(x) = x_1x_2f_0(x_1^2, x_2^2, \ldots)$). This completes the calculation of the L^2 norms of the harmonic polynomials. The ${}_4F_3$ series allow no further simplification.

5. Discussion

To conclude, we discuss the significance of the results, especially with regard to applications and indications for further research. The problem that was solved here is, in a sense, the minimal approach to constructing harmonic polynomials of type B. It may turn out that a different normalization may be more useful or concise; for example, the value $h_{4n,0}(1,0,\ldots)$ can be written as $\frac{(-1)^n(k+1)_n\Lambda(2n,2n)}{n!\Lambda(4n,0)}$ and similar expressions hold for the other formulae in section 4.3. The expression for $\|h_{4n,0}\|^2$ is also somewhat simplified by changing the normalization to $\frac{\Lambda(4n,0)}{\Lambda(2n,2n)}h_{4n,0}$. Of course the ${}_4F_3$ part stays.

5.1. Application

There is a quantum many-body exactly-solvable model associated with Δ_B, namely the spin Calogero model of Yamamoto and Tsuchiya [10, 11]. This deals with N identical particles on a line with inverse-square mutual repulsion potential and an external harmonic confinement potential. In addition, the particles have a two-valued spin which can be exchanged between them. The construction of eigenfunctions in terms of nonsymmetric Jack and generalized Hermite polynomials was discussed in [7]. The Hamiltonian for the system (with $\omega, k, k_1 > 0$) is

$$\mathcal{H} = \sum_{i=1}^{N}\left\{-\left(\frac{\partial}{\partial x_i}\right)^2 + \omega^2 x_i^2 + \frac{k_1(k_1-\sigma_i)}{x_i^2}\right\} + 2k\sum_{1\leqslant i<j\leqslant N}\left\{\frac{k-\sigma_{ij}}{(x_i-x_j)^2} + \frac{k-\tau_{ij}}{(x_i+x_j)^2}\right\}.$$

The ground state for the system is

$$\psi(x) = \prod_{i=1}^{N}|x_i|^{k_1}\prod_{1\leqslant i<j\leqslant N}|x_i^2-x_j^2|^k \exp\left(-\frac{\omega|x|^2}{2}\right).$$

Then the conjugate $\psi\mathcal{H}\psi^{-1} = 2\omega(\sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} + Nk_2) - \Delta_B$. Let $f_m(x)$ be a harmonic and homogeneous polynomial of degree m, then for $n = 0, 1, 2, \ldots$ the function $L_n^{(c)}(\omega|x|^2) f_m(x)\psi(x)$ is an eigenfunction of $\mathcal{H}$ with eigenvalue $2\omega(m + 2n + Nk_2)$, where $c = m + Nk_2 - 1$. Here

$$L_n^{(c)}(t) = \frac{(c+1)_n}{n!}\sum_{i=0}^{n}\frac{(-n)_i}{(c+1)_i}\frac{t^i}{i!}$$

denotes the Laguerre polynomial of index c and degree n. The set of all such functions with $m + 2n = s$ spans all the eigenfunctions with eigenvalue $2\omega(s + Nk_2)$. The set $\{L_n^{(c)}(\omega|x|^2) f_m(x)\}$ was used as a basis for polynomials in the study of inner products [3] and the Hankel transform [4]. Van Diejen [1] considered W_N -invariant eigenfunctions of this type for $\mathcal{H}$. The polynomials $h_{2n,0}$ can produce such invariants by summing over translates:

$$\left(1 + \sum_{j>2}\sigma_{2j} + \sum_{2<i<j\leqslant N}\sigma_{1i}\sigma_{2j}\right)h_{2n,0}(x).$$

5.2. Further work

It is still an open problem to find an orthogonal basis for the harmonic homogeneous polynomials. Such bases are useful in approximation theory and numerical cubature (see Xu [8, 9]). It is not difficult to write down self-adjoint operators on polynomials, for example $(x_i T_j - x_j T_i)^2$ for $1 \leqslant i < j \leqslant N$. This is a useful method for Abelian reflection groups. However, computer algebra calculations reveal that the characteristic polynomials of these operators on polynomials of not large degree do not factor linearly in $\mathbb{Q}(k, k_1)$ (for type B). Hence, one does not expect tractable eigenfunction decompositions. It seems worthwhile to try to extend to more variables the generating function construction for F_0, F_1 which was the main device for this paper (that is, consider harmonic polynomials annihilated by T_i for $i > n_0$; already the case $n_0 = 3$ is interesting). Obviously, a more sophisticated way of handling the different cases will need to be developed. The present approach is just tolerable for the different basis functions involved in the representation theory of B_2. It certainly seems that finding orthogonal bases is considerably more complicated then the construction of nonsymmetric Jack polynomials.

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